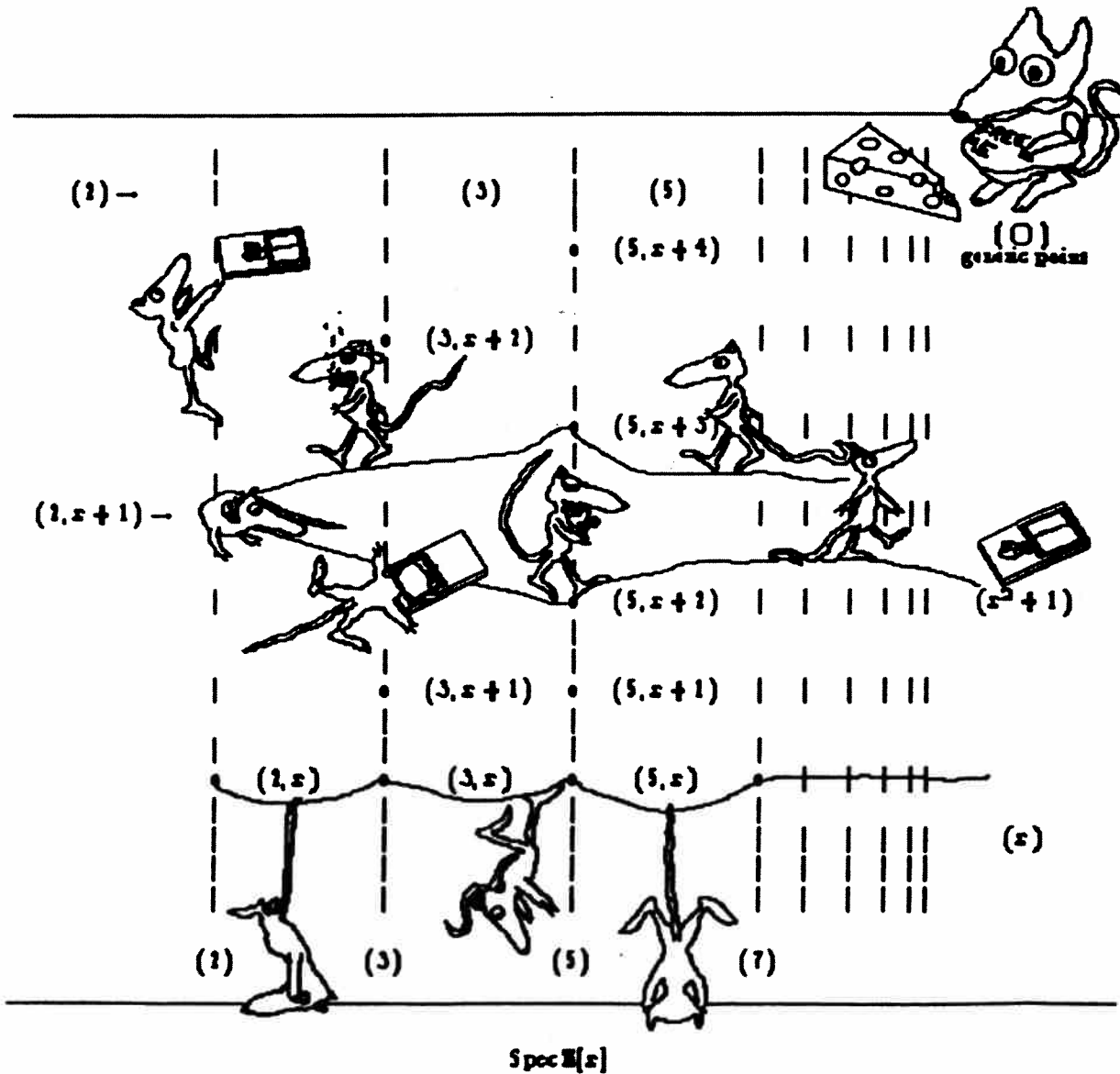


ALGEBRAIC GEOMETRY

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INTRODUCTION TO ALGEBRAIC GEOMETRY

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ABSTRACT. Algebraic geometry is a mixture of the ideas of two Mediterranean cultures. It is the superposition of the Arab science of the lightening calculation of the solutions of equations over the Greek art of position and shape. This tapestry was originally woven on European soil and is still being refined under the influence of international fashion. Algebraic geometry studies the delicate balance between the geometrically plausible and the algebraic possible. Whenever one side of this mathematical teeter-totter outweighs the other, one immediately loses interest and runs off in search of a more exciting amusement.

George R. Kempf
1944 - 2002

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1. ALGEBRA PREREQUISITES

Students should be familiar with certain basic ideas in commutative algebra, such as the first few pages of Atiyah-Macdonald, *Introduction to Commutative Algebra*. For example, commutative rings \mathbb{A} , ideals, \mathbb{A} -modules and \mathbb{A} -algebras, tensor products, integral domains, and unique factorization domains (UFD). From field theory, one should know the meaning of algebraic extensions, algebraic closure, algebraically independent sets, transcendence base and degree (some knowledge of the relationship between transcendence degree of finitely generated separable field extensions and derivations would be helpful, but not required).

All rings \mathbb{A} are assumed commutative with unity $1 \in \mathbb{A}$. We sometimes write $\mathbb{A} = (1)$. Any homomorphism of rings $f : \mathbb{A} \rightarrow \mathbb{B}$ must satisfy $f(1) = 1$. (By definition, such a homomorphism f turns \mathbb{B} into an \mathbb{A} -algebra.) We recall the following.

Definition 1.1. (i) An ideal $\wp \subset \mathbb{A}$ is said to be prime if $\wp \neq (1)$ and the following equivalent conditions hold:

- $\forall a, b \in \mathbb{A}, ab \in \wp \Rightarrow a \in \wp$ or $b \in \wp$.

Date: May 25, 2009.

- \mathbb{A}/\wp is an integral domain.

(ii) An ideal $\mathcal{M} \subset \mathbb{A}$ is said to be maximal if $\mathcal{M} \neq (1)$ and the following equivalent conditions hold:

- If $\mathcal{U} \subset \mathbb{A}$ is an ideal such that $\mathcal{M} \subseteq \mathcal{U}$, then either $\mathcal{U} = \mathcal{M}$ or $\mathcal{U} = (1)$.
- \mathbb{A}/\mathcal{M} is a field.

Remark 1.2. A Zorn’s lemma argument shows that any ring \mathbb{A} contains a maximal ideal. [Exercise. Fill in the details of this, as well as the proofs of the equivalent statements in the definition above.]

2. MOTIVATION

Consider the polynomial

$$f(z, w) = 2ew^2 - \pi z^3 + \sqrt{\pi}z + 2\sqrt{-1} \in \mathbb{C}[z, w],$$

and the solution set

$$X := V(f) := \{p \in \mathbb{C}^2 \mid f(p) = 0\}.$$

Calculating the gradient leads to a vector “perpendicular” to X . We have

$$\nabla f = (-3\pi z^2 + \sqrt{\pi}, 4ew) = (0, 0) \Leftrightarrow (z, w) = \left(\pm \frac{1}{\sqrt{3\sqrt{\pi}}}, 0 \right)$$

In particular $\nabla f(p) = (0, 0) \Rightarrow p \in \mathbb{R}^2$, hence $f(p) \neq 0$ (otherwise $\sqrt{-1} \in \mathbb{R}$). Since $\nabla f(p) \neq (0, 0) \forall p \in X$, one can argue from the point of view of analysis that X is locally a disk in the strong topology on \mathbb{C}^2 (implicit function theorem), i.e. a complex (open) manifold of dimension 1. X is an example of an open Riemann surface, and a very special case of a Stein manifold. Its tangent space at any $p \in X$ is given by $T_p(X) = \{p + v \in \mathbb{C}^2 \mid v \bullet \nabla f(p) = 0\} \simeq \mathbb{C}$. From the point of view of a complex geometer, X has a complex structure with an underlying differentiable structure $X_{\mathbb{R}}$, and it would be of interest to study the family of complex structures with underlying $X_{\mathbb{R}}$ (a real orientable surface). Put differently, we would like to think of X as part of a family of Riemann surfaces, and we will comment on this shortly.

As an algebraist, one can study the ring

$$R_X := \frac{\mathbb{C}[z, w]}{(f(z, w))}.$$

Then X can be identified with the \mathbb{C} -algebra homomorphisms

$$X = \text{Alg}_{\mathbb{C}}(R_X, \mathbb{C}).$$

Every such homomorphism $\psi \in \text{Alg}_{\mathbb{C}}(R_X, \mathbb{C})$ has $\ker \psi =$ a maximal ideal of R_X . The maximal ideals are of the form $\mathcal{M}_p := (z - p_1, w - p_2) \subset R_X$, where $p = (p_1, p_2) \in X$. So one can interpret $X = \text{Max}(R_X)$, the “space” of maximal ideals (maximal spectrum).

An algebraic geometer make take this one step further. Instead of R_X , create a new algebra over a number field that encodes all the information about X , as part of

a family of Riemann surfaces, and which incorporates some arithmetic information about X . Consider the number field $K := \mathbb{Q}[\sqrt{-1}]$, and replace R_X by

$$R_Y := \frac{K[z, w, u, v, t]}{(2tw^2 - uz^3 + vz + 2\sqrt{-1}, u - v^2)}.$$

Now put

$$R_S := \frac{K[u, v, t]}{(u - v^2)} \simeq K[v, t],$$

with quotient field

$$K(S) := \text{Quot}(R_S).$$

An embedding $K(S) \hookrightarrow \mathbb{C}$ corresponds to a general complex point in $S := V(u - v^2) \subset \mathbb{C}^3$, via $R_S \hookrightarrow K(S) \hookrightarrow \mathbb{C}$. For example, the assignment of $(u, v, t) = (\pi, \sqrt{\pi}, e)$ determines an embedding. In particular the points in S parameterize a family of open Riemann surfaces, where X corresponds to $(\pi, \sqrt{\pi}, e) \in S$. One has an inclusion $R_S \hookrightarrow R_Y$ inducing a map of “complex” spaces

$$h : Y_{\mathbb{C}} := \text{Max}(R_Y \otimes_K \mathbb{C}) \rightarrow S_{\mathbb{C}} := \text{Max}(R_S \otimes_K \mathbb{C}),$$

where $R_Y \otimes_K \mathbb{C}$ amounts to changing K in R_Y by \mathbb{C} (same for $R_S \otimes_K \mathbb{C}$). Here we have $h^{-1}(\pi, \sqrt{\pi}, e) = X$. Intuitively, $\dim S = 2$ and $\dim Y = 3$. The fiber dimension of h is $3 - 2 = 1$.

An arithmetic geometer is inclined to dig further. Note that a choice of $\sqrt{-1}$ being a solution of $x^2 + 1 = 0$ enables us to encode everything into what is called a proper model over \mathbb{Z} . Namely, put

$$R_{Y, \mathbb{Z}} := \frac{\mathbb{Z}[z, w, u, v, t, x]}{(2tw^2 - uz^3 + vz + 2x, u - v^2, x^2 + 1)}.$$

One has $\mathbb{Z} \hookrightarrow R_{Y, \mathbb{Z}}$ and instead of maximal ideals, one considers prime ideal spaces (prime spectrum):

$$Y_{\mathbb{Z}} := \text{Spec}(R_{Y, \mathbb{Z}}) \rightarrow \text{Spec}(\mathbb{Z}).$$

As $\dim \text{Spec}(\mathbb{Z}) = 1$, $\dim Y_{\mathbb{Z}} = 4$. For example, over the prime ideal $(0) \subset \mathbb{Z}$, we are really studying $Y_{\mathbb{Z}}$ as a complex space, where over $(p) \subset \mathbb{Z}$ we can work modulo p . This leads to the concept of reduction mod p . For example, over \mathbb{Z}_2 , we are studying

$$\frac{\mathbb{Z}_2[z, w, u, v, t, x]}{(-uz^3 + vz, u - v^2, x^2 + 1)}.$$

So in summary, the basic object we considered, after compactifying, is an elliptic curve. To a topologist, an elliptic curve is a torus. To a complex geometer, it is a torus with a family of complex structures, represented by a family over a surface S . To an algebraic geometer, there is arithmetical information that comes equipped with all of this, leading to complexity of points (heights), Frobenius actions (mod p), and so forth.

3. CLASSICAL STORY I

Let k be a field. and $K \supset k$ an overfield. Denote by $k[X_1, \dots, X_n]$ the polynomial algebra in n letters. As mentioned earlier, all rings that we will consider will be commutative with the identity element 1. And to re-iterate, that all ring (resp. algebra) homomorphisms map 1 to 1.

Definition 3.1. An affine closed algebraic set $V \subset K^n$ is the zeros of a finite set of polynomials $f_1, \dots, f_m \in k[X_1, \dots, X_n]$. It is often written as $V = V(f_1, \dots, f_m)$; i.e V is cut out by the zeros of $f_j = 0$ for $j = 1, \dots, m$, i.e $\{\xi \in K^n \mid f_1(\xi) = \dots = f_m(\xi) = 0\}$.

Exercise. Consider the vector space k^{n^2} of $n \times n$ matrices. Show that

$$X := \{A \in k^{n^2} \mid \det(A) \neq 0\},$$

is an affine closed algebraic set, where $K = k$.

Remark 3.2. $V(f_1, \dots, f_m) = V(\mathcal{U}) := \{\xi \in K^n \mid f(\xi) = 0 \forall f \in \mathcal{U}\}$ where $\mathcal{U} = (f_1, \dots, f_m) \subset k[X_1, \dots, X_n]$ is the ideal generated by f_1, \dots, f_m .

Recall that the ring \mathbb{A} is called Noetherian if it does *not* admit an infinitely ascending chain of ideals,

$$\mathcal{U}_1 \subsetneq \mathcal{U}_2 \subsetneq \mathcal{U}_3 \subsetneq \dots \subsetneq \mathcal{U}_m \subsetneq \mathcal{U}_{m+1} \subsetneq \dots;$$

equivalently every ideal of \mathbb{A} is finitely generated (see the Appendix for the proof). If \mathbb{A} is Noetherian then Hilbert Basis theorem says that $\mathbb{A}[t]$ is also Noetherian, for an indeterminate t . Hence we arrive at:

Corollary 3.3. $k[X_1, \dots, X_n]$ is Noetherian.

Proof. k has only two ideals: 0 and itself $k = (1)$, hence is obviously Noetherian. Now apply Hilbert Basis Theorem and induction on n using

$$k[X_1, \dots, X_n] = (k[X_1, \dots, X_{n-1}])[X_n].$$

□

Let $\mathcal{U} \subset k[X_1, \dots, X_n]$ be an ideal. Then $\mathcal{U} = (f_1, \dots, f_m)$ for some f_1, \dots, f_m by Hilbert Basis Theorem.

Definition 3.4.

$$k[x_1, \dots, x_n] := \frac{k[X_1, \dots, X_n]}{\mathcal{U}},$$

is called a k -algebra of finite type.

Remark 3.5. Since ascending chains of ideals pullback to ascending chains of ideals under $k[X_1, \dots, X_n] \rightarrow k[x_1, \dots, x_n]$, it is obvious that $k[x_1, \dots, x_n]$ is Noetherian (an exercise!). Furthermore, it is clear that

$$V(\mathcal{U}) = \text{Alg}_k(k[x_1, \dots, x_n], K).$$

Proposition 3.6. (i) $\mathcal{U}_1 \subseteq \mathcal{U}_2 \implies V(\mathcal{U}_2) \subseteq V(\mathcal{U}_1)$.

(ii) $V(\mathcal{U}_1) \cup V(\mathcal{U}_2) = V(\mathcal{U}_1 \cap \mathcal{U}_2) = V(\mathcal{U}_1 \cdot \mathcal{U}_2)$, where

$$\mathcal{U}_1 \cdot \mathcal{U}_2 = \left\{ \sum_{\alpha \in I} f_{1\alpha} f_{2\alpha}, f_{i\alpha} \in \mathcal{U}_i \right\}.$$

(iii) $V(\sum_{\alpha \in I} \mathcal{U}_\alpha) = \bigcap_{\alpha \in I} V(\mathcal{U}_\alpha)$ where I is any index set and

$$\sum_{\alpha \in I} \mathcal{U}_\alpha = \left\{ \sum_{\alpha \in J} f_\alpha \mid J \text{ finite in } I \right\}.$$

(iv) $V(0) = K^n$, $V(1) = \emptyset$.

(v) Let $p = (p_1, \dots, p_n) \in k^n$ and $\mathcal{M}_p = (X_1 - p_1, \dots, X_n - p_n)$ (a maximal ideal). Then $V(\mathcal{M}_p) = \{p\}$.

(vi) Let $\sqrt{\mathcal{U}} = \{f \in k[X_1, \dots, X_n] \mid f^m \in \mathcal{U} \text{ for some } m = m(f) \in \mathbb{N}\} = \text{radical of } \mathcal{U}$. Then $V(\sqrt{\mathcal{U}}) = V(\mathcal{U})$.

Proof. Let us prove e.g. the property (ii). Note that

$$\mathcal{U}_1 \cap \mathcal{U}_2 \supseteq \mathcal{U}_1 \cdot \mathcal{U}_2,$$

so by property (i) we get

$$V(\mathcal{U}_1 \cap \mathcal{U}_2) \subseteq V(\mathcal{U}_1 \cdot \mathcal{U}_2).$$

Also we have

$$V(\mathcal{U}_1) \cup V(\mathcal{U}_2) \subseteq V(\mathcal{U}_1 \cap \mathcal{U}_2).$$

Now let us show the other inclusion. Let $\xi \in V(\mathcal{U}_1 \cdot \mathcal{U}_2)$ and suppose $\xi \notin V(\mathcal{U}_1)$ then $\exists f_1 \in \mathcal{U}_1$ such that $f_1(\xi) \neq 0$. By our choice of ξ we have $\forall f_2 \in \mathcal{U}_2$ $f_1(\xi)f_2(\xi) = 0$, hence $f_2(\xi) = 0 \forall f_2 \in \mathcal{U}_2$. So $\xi \in V(\mathcal{U}_2)$ which proves the property (ii). \square

Definition 3.7. Properties (i) - (iv) implies that (Zariski) closed subsets of K^n form a topology of closed sets in K^n called the Zariski topology. Equivalently, a Zariski open subset is given by $U = K^n \setminus V$, where V is Zariski closed.

Remark 3.8. (1) For any Zariski closed subset $W \subset K^n$, one has the induced Zariski topology on W .

(2) A basis of open sets for the Zariski topology can be arrived at as follows: Let $W = V(\mathcal{U}_0) \subset K^n$, $\mathcal{U}_0 \subset k[X_1, \dots, X_n]$ an ideal, be a Zariski closed set and let $\mathcal{U} \subset k[X_1, \dots, X_n]$ be another ideal where $\mathcal{U}_0 \subset \mathcal{U}$. Put $E = V(\mathcal{U}) \subset W$. Note that \mathcal{U} is finitely generated, so $\mathcal{U} = (f_1, \dots, f_r)$, hence $E = V(f_1, \dots, f_r) = V(f_1) \cap \dots \cap V(f_r)$. We have $W \setminus E$ is open and $W \setminus E = W \setminus \{V(f_1) \cap \dots \cap V(f_r)\} = \{W \setminus V(f_1)\} \cup \dots \cup \{W \setminus V(f_r)\}$. For $f \in R_W := k[X_0, \dots, X_n]/\mathcal{U}_0$ set $W_f = W \setminus V(f)$. Hence $\{W_f \mid f \in R_W\}$ form a basis of open sets for the Zariski topology of W .

(4) In the case $K = \mathbb{C}$, it is known that any non-empty Zariski open set $U \subset \mathbb{C}^n$ is a dense subset of \mathbb{C}^n , in the strong topology.

Now recall that an ideal $\wp \subset \mathbb{A}$ is prime if $\wp \neq (1) := \mathbb{A}$ and \mathbb{A}/\wp is an integral domain, and that for any ideal $\mathcal{U} \subset \mathbb{A}$, the radical is given by $\sqrt{\mathcal{U}} := \{a \in \mathbb{A} \mid a^m \in \mathcal{U}, \text{ for some } m = m(a) \in \mathbb{N}\}$.

Proposition 3.9. (i) For a prime ideal $\wp \subset \mathbb{A}$, $\sqrt{\wp} = \wp$.

(ii) For any ideal $\mathcal{U} \subset \mathbb{A}$,

$$\sqrt{\mathcal{U}} = \bigcap_{\wp \supset \mathcal{U}} \wp.$$

Proof. Part (i) is obvious. The proof of part (ii) is in the Appendix. \square

Exercise. Give a direct proof that $\sqrt{\mathcal{U}} \subset \mathbb{A}$ is an ideal.

Definition 3.10. Let $V = V(\mathcal{U}) \subset K^n$. The ideal of polynomials vanishing on V is given by

$$I(V) = \{f \in k[X_1, \dots, X_n] \mid f \equiv 0 \text{ on } V\}.$$

[Note. Observe that $I(\emptyset) = k[X_1, \dots, X_n]$.]

Corollary 3.11 (to the definition). *If $V = V(\mathcal{U})$, then*

- (i) $\sqrt{\mathcal{U}} \subseteq I(V)$,
- (ii) $V = V(I(V))$,
- (iii) $\sqrt{I(V)} = I(V)$.
- (iv) $I(V(I(V))) = I(V)$ (*Immediate from (ii).*)

4. CLASSICAL STORY II

The inclusion in statement (i) in the above corollary can be replaced by an equality if $K = \bar{k}$ (= algebraic closure of k). This is a consequence of the Hilbert Nullstellensatz theorem. It is also the case when $K = \Omega$, where Ω is a universal overfield.

Definition 4.1. Let $\Omega \supset k$ be an overfield. Then Ω is universal over k if it is algebraically closed and has infinite transcendence degree over k .

The utility of universality can be seen as follows. Let us assume that $K = \Omega$ is universal over k . Let $\mathcal{U} \subset k[X_1, \dots, X_n]$ be an ideal, $X = V(\mathcal{U}) \subset \Omega^n$, and

$$R_X = \frac{k[X_1, \dots, X_n]}{\mathcal{U}}.$$

Note the identification:

$$X = \text{Alg}_k(R_X, \Omega).$$

Further, any $\psi \in \text{Alg}_k(R_X, \Omega)$ determines a prime ideal $\wp := \ker \psi \subset R_X$. Conversely, let $\wp \subset R_X$ be a prime ideal, and put $L = \text{Quot}(R_X/\wp)$. Then L/k is a field extension of finite transcendence degree over k . The universality of Ω guarantees the existence of an embedding $L \hookrightarrow \Omega$ of fields over k . Now the composite

$$R_X \twoheadrightarrow R_X/\wp \hookrightarrow L \hookrightarrow \Omega,$$

defines an element of $\text{Alg}_k(R_X, \Omega)$. Thus one can think of the prime spectrum $\text{Spec}(R_X)$ as defining an equivalence class of “solutions” in $V(\mathcal{U})$, namely for $\psi_1, \psi_2 \in \text{Alg}_k(R_X, \Omega)$, $\psi_1 \equiv \psi_2 \Leftrightarrow \ker \psi_1 = \ker \psi_2$. Note that we can regard $\bar{k} \subset \Omega$. A solution $\psi \in \text{Alg}_k(R_X, \Omega)$ is said to be algebraic if $\psi(R_X) \subset \bar{k}$. A version of the Hilbert Nullstellensatz says that if B is a k -algebra of finite type, then B is a field iff B is algebraic, viz., $B \hookrightarrow_{/k} \bar{k}$, equivalently, every element of B is a root of a polynomial in $k[t]$. In particular, if $\mathcal{M} \subset R_X$ is a maximal ideal, then $B := R_X/\mathcal{M}$ is a field, hence algebraic. Thus $\mathcal{M} = \ker \psi$ for some $\psi \in \text{Alg}_k(R_X, \bar{k})$. Thus the algebraic solutions correspond in the same way to the maximal ideals in R_X .

Proposition 4.2. *For $K = \Omega$ universal, $I(V(\mathcal{U})) = \sqrt{\mathcal{U}}$.*

Proof. We need only show the inclusion $I(V(\mathcal{U})) \subseteq \sqrt{\mathcal{U}}$. Let $\wp \supset \mathcal{U}$ be a prime ideal, with image $\bar{\wp} \subset R_X$, which is likewise prime in R_X (why?). By the above discussion, $\exists \psi \in \text{Alg}_k(R_X, \Omega)$ such that $\bar{\wp} = \ker \psi$. If $\pi : k[X_1, \dots, X_n] \twoheadrightarrow R_X$ is the quotient homomorphism, then $\pi^{-1}(\bar{\wp}) = \wp$, and hence $\wp = \ker(\psi \circ \pi)$. Now let

$p_j = \psi \circ \pi(X_j)$. Then $p := (p_1, \dots, p_n) \in V(\mathcal{U})$. Now suppose that $f \in I(V(\mathcal{U}))$. Then $f \equiv 0$ on $V(\mathcal{U}) \Rightarrow f(p) = 0$, hence $f \in \wp$. We have in fact shown that

$$f \in \bigcap_{\wp \supset \mathcal{U}} \wp (= \sqrt{\mathcal{U}}),$$

a fortiori $I(V(\mathcal{U})) \subseteq \sqrt{\mathcal{U}}$. □

Corollary 4.3. *Suppose $k = K = \mathbb{C}$. Then $I(V(\mathcal{U})) = \sqrt{\mathcal{U}}$.*

Proof. Let $X = V(\mathcal{U}) \subset \mathbb{C}^n$, and $f \in I(X)$ be given. By Hilbert's basis theorem,

$$\mathcal{U} = (f_1, \dots, f_r) \subset \mathbb{C}[X_1, \dots, X_n],$$

is finitely generated. For the moment, let k be the subfield of \mathbb{C} made up of \mathbb{Q} adjoined with the coefficients $\{c_1, \dots, c_M\}$ of $\{f, f_1, \dots, f_r\}$, i.e. $k = \mathbb{Q}(c_1, \dots, c_M)$. Then \mathbb{C}/k is a universal overfield. Thinking of now $\{f_1, \dots, f_r\} \subset k[X_1, \dots, X_n]$, we set

$$\mathcal{U}_0 = (f_1, \dots, f_r) \subset k[X_1, \dots, X_n],$$

and accordingly $V(\mathcal{U}_0) = X$. By the previous proposition, $f \in I(V(\mathcal{U}_0)) = \sqrt{\mathcal{U}_0}$, and hence via the inclusion $\sqrt{\mathcal{U}_0} \subset \sqrt{\mathcal{U}}$, $f \in \sqrt{\mathcal{U}}$. We have thus shown that $I(V(\mathcal{U})) \subseteq \sqrt{\mathcal{U}}$, and hence equality holds. □

Definition 4.4. $V \neq \emptyset$ is irreducible if $V \neq V_1 \cup V_2$, where V_1, V_2 closed algebraic and $V_i \not\subseteq V_j$ for $i \neq j$.

Proposition 4.5. V is irreducible $\iff I(V) = \wp$ prime.

Proof. (\implies): Set $\mathcal{U} = I(V)$ ($\Rightarrow V = V(\mathcal{U})$). Assume to contrary \mathcal{U} is not prime. Thus $\exists f, g \in k[X_1, \dots, X_n]$, $f, g \notin \mathcal{U}$ such that $f \cdot g \in \mathcal{U}$. Then

$$V = V(\mathcal{U}) \subset V(f \cdot g) = V(f) \cup V(g).$$

Now set $V_1 = V(f) \cap V$, $V_2 = V(g) \cap V$. Note that $V_1 = V \Rightarrow f \in I(V_1) = I(V) = \mathcal{U}$, a contradiction. Hence $V_1 \neq V$. Similarly $V_2 \neq V$.

(\impliedby): Suppose $I(V) = \wp$ prime but that $V = V_1 \cup V_2$. Then $V_1 = V(\mathcal{U}_1)$, $V_2 = V(\mathcal{U}_2)$ and $V(\wp) = V(\mathcal{U}_1 \cdot \mathcal{U}_2) \Rightarrow \mathcal{U}_1 \cdot \mathcal{U}_2 \subset I(V) = \wp$. If $\mathcal{U}_1 \not\subseteq \wp$ then $\exists f \in \mathcal{U}_1 \setminus \wp$ with $f \cdot \mathcal{U}_2 \subset \wp$. But \wp prime implies $\mathcal{U}_2 \subset \wp$, so $V_2 \supseteq V \Rightarrow V = V_2$. □

In summary, we have the following picture for \mathbb{A} a k -algebra of finite type, with Ω/k universal and $\bar{k} \subset \Omega$: ($\text{Spec}(\mathbb{A}) = \{\text{prime ideals in } \mathbb{A}\}$)

$$\begin{array}{ccc} \text{Alg}_k(\mathbb{A}, \bar{k}) & \hookrightarrow & \text{Alg}_k(\mathbb{A}, \Omega) \\ \downarrow \text{"ker"} & & \downarrow \text{"ker"} \\ \text{Max}(\mathbb{A}) & \hookrightarrow & \text{Spec}(\mathbb{A}) \end{array}$$

By universality of Ω , the second vertical arrow is surjective. By Hilbert, the first vertical arrow is likewise surjective (and in fact a bijection if $k = \bar{k}$). [The proof injectivity in the bijection goes as follows: Let $\psi_1, \psi_2 \in \text{Alg}_k(\mathbb{A}, k)$ with $\psi_1 \neq \psi_2$. Then $\exists \xi \in \mathbb{A}$ such that $\psi_1(\xi) \neq \psi_2(\xi)$. But $\psi_2(\xi) \in \bar{k} = k \Rightarrow \psi_j(\psi_2(\xi)) = \psi_2(\xi)$. Thus $\xi - \psi_2(\xi) \in \ker \psi_2$, whereas $\xi - \psi_2(\xi) \notin \ker \psi_1$. Thus $\ker \psi_1 \neq \ker \psi_2$ define two different maximal deals in \mathbb{A} .]

5. CLASSICAL STORY III

We will now consider the situation where $K = \bar{k}$.

Theorem 5.1 (Hilbert Nullstellensatz). *If \wp is a prime ideal, then \wp is precisely the ideal of polynomials $f \in k[X_1, \dots, X_n]$ that vanish identically on $V(\wp) \subset \bar{k}^n$, i.e. $\wp = I(V(\wp))$. More generally, if \mathcal{U} is any ideal, then $\sqrt{\mathcal{U}} = I(V(\mathcal{U}))$.*

Proof. See the appendix for the proof. \square

Exercise. Consider the ideal $\mathcal{U} = (x^2 + 1) \subset \mathbb{R}[x]$, with $k = K = \mathbb{R}$. Show that $\sqrt{\mathcal{U}} \neq I(V(\mathcal{U}))$.

Corollary 5.2. *Suppose that $k = \bar{k}$ is algebraically closed. Every maximal ideal in $k[X_1, \dots, X_n]$ of the form $\mathcal{M}_p = (X_1 - p_1, \dots, X_n - p_n)$ where $p \in k^n$.*

Proof. Let \mathcal{M} be a maximal ideal. By Nullstellensatz $V(\mathcal{M}) \neq \emptyset$, otherwise $(1) = \sqrt{\mathcal{M}} = \mathcal{M}$. Choose $p = (p_1, \dots, p_n) \in V(\mathcal{M})$. Then $\{p\} = V(\mathcal{M}_p) \subseteq V(\mathcal{M}) \Rightarrow \mathcal{M} = I(V(\mathcal{M})) \subseteq I(V(\mathcal{M}_p)) = \mathcal{M}_p$. By maximality, $\mathcal{M} = \mathcal{M}_p$. \square

Corollary 5.3. *Suppose $k = K = \bar{k}$, and $\mathcal{U} \subset k[X_1, \dots, X_n]$ an ideal. Put $X = V(\mathcal{U})$. Then*

$$X \simeq \text{Max}(R_X).$$

Proof. Exercise! \square

Now suppose that $K = \bar{k}$, $\mathcal{U} \subset k[X_1, \dots, X_n]$ an ideal, $X = V(\mathcal{U})$, with $R_X = k[X_1, \dots, X_n]/\mathcal{U}$. For $f \in R_X$, put $X_f = X \setminus V(f)$.

Exercise. Show that X_f is an affine closed algebraic set with coordinate ring $R_{X_f} = R_X[t]/(t \cdot f - 1)$.

Recall that $\{X_f \mid f \in R_X\}$ forms a basis of open sets for the Zariski topology of X . Then one can also prove the following quasi-compactness property:

Proposition 5.4. *Let $\{f_\alpha\}_{\alpha \in I} \subset R_X$ be any collection satisfying*

$$X = \bigcup_{\alpha \in I} X_{f_\alpha}.$$

Then

$$X = X_{f_1} \cup \dots \cup X_{f_N},$$

for some finite sub-collection $\{f_1, \dots, f_N\} \subset \{f_\alpha\}_{\alpha \in I}$.

Proof. We have

$$\emptyset = X \setminus \left\{ \bigcup_{\alpha \in I} X_{f_\alpha} \right\} = V\left(\mathcal{U} + \sum_{\alpha \in I} (f_\alpha)\right).$$

By the Nullstellensatz,

$$\mathcal{U} + \sum_{\alpha \in I} (f_\alpha) = (1) = k[X_1, \dots, X_n],$$

hence $\exists g_1, \dots, g_N \in R_X$ such that

$$\sum_{j=1}^N g_j f_j = 1 \in R_X.$$

Hence for any $p \in X$, $f_j(p) \neq 0$ for some j . \square

From now on, we will assume that $K = k = \bar{k}$.

Definition 5.5. A closed algebraic set $V(\wp)$ where $\wp \subset k[X_1, \dots, X_n]$ is a prime ideal, is called an affine algebraic variety.

Corollary 5.6. *There is an order reversing bijection*

$$\left\{ \begin{array}{l} \text{Ideals } \mathcal{U} \subset \\ k[X_1, \dots, X_n] \\ \mathcal{U} = \sqrt{\mathcal{U}} \end{array} \right\} \begin{array}{c} \xrightarrow{V} \\ \xleftarrow{I} \end{array} \left\{ \begin{array}{l} \text{Algebraic} \\ \text{subsets} \\ X \subset k^n \end{array} \right\}$$

where the prime ideals correspond to varieties.

Lemma 5.7. *Let X be a variety and $U \subset X$ be non-empty Zariski open set. Then $\overline{U}^{\text{Zar}} = X$*

Proof. Set $V = \overline{U}^{\text{Zar}}$ and $W = X \setminus U \subset X$. Then $X = W \cup V$. X is irreducible so $X \subseteq W$ or $X \subseteq V$. If $X \subseteq W$ then $W = X$. But this contradicts $U \neq \emptyset$. So $X \subseteq V$. But also $V \subseteq X$, hence $X = V$. \square

Definition 5.8. Let $X = V(\wp) \subset k^n$ be a variety.

$$R_X := k[X_1, \dots, X_n]/\wp,$$

is called the affine coordinate ring associated to X . It is precisely the functions on X which are the restrictions to X of polynomials on k^n .

Proposition 5.9. *Let $X = V(\mathcal{U}) \subset k^n$ be an algebraic set. Then \exists varieties $V_1, \dots, V_N \subset k^n$ such that*

$$X = V_1 \cup \dots \cup V_N.$$

If we assume moreover that $V_i \not\subseteq V_j$ for all $i \neq j$, then (up to relabelling), this decomposition is unique.

Proof. (Outline.) First of all, if we have two irreducible decompositions:

$$V_1 \cup \dots \cup V_N = X = W_1 \cup \dots \cup W_M,$$

then e.g.:

$$V_1 = \{V_1 \cap W_1\} \cup \dots \cup \{V_1 \cap W_M\}.$$

Hence up to relabelling, $V_1 \subseteq W_1$. But $W_1 \subseteq V_j$ by the same reasoning, and hence $1 = j$. Thus the real issue is existence. If X is irreducible (viz., a variety), then we are done. Otherwise

$$X = X_1 \cup X_2,$$

where $X_1 \neq X_2$ are algebraic. If X_1 and X_2 are both irreducible then we are done. Otherwise we end up with

$$I(X) \subsetneq I(X_j) \subsetneq I(X_{j_1}) \subsetneq \dots,$$

which must terminate by the ascending chain condition on $k[X_1, \dots, X_n]$. \square

Example. Let $f \in k[X_1, \dots, X_n]$ be a non-unit. Then since $k[X_1, \dots, X_n]$ is a UFD, one has a unique decomposition of f into irreducibles, viz.,

$$f = \prod_{j=1}^N f_j^{k_j}.$$

Note that (f_j) is a prime ideal for all j . Put $V_j = V(f_j)$. Then

$$X = V_1 \cup \cdots \cup V_N.$$

Exercise. Let $h(x, y)$ be a homogeneous polynomial of degree $d \geq 1$. Show that $V(h) \subset k^2$ is a union of $\leq d$ lines.

Exercise. Decompose $X := V(x^2 - yz, xz - x) \subset k^3$ into a union of irreducible varieties.

Let $X = V(\mathcal{U}_1) \subset k^n$, $Y = V(\mathcal{U}_2) \subset k^m$ be algebraic sets, with corresponding R_X and R_Y . Let $h \in \text{Alg}_k(R_Y, R_X)$. For $\psi \in \text{Alg}_k(R_X, k) = X$, we have $\psi \circ h \in \text{Alg}_k(R_Y, k) = Y$. I.e. h induces a map $T_h : X \rightarrow Y$. Such a map is called a morphism from X to Y . If $p \in X$, then $h^{-1}(\mathcal{M}_p) = \mathcal{M}_q$ for some $q \in Y$. If we write $R_X = k[x_1, \dots, x_n]$, $R_Y = k[y_1, \dots, y_m]$, and if we write $g_j(x_1, \dots, x_n) = h(y_j) \in k[x_1, \dots, x_n]$, Then T_h is induced by

$$(Y_1, \dots, Y_m) = (g_1(X_1, \dots, X_n), \dots, g_m(X_1, \dots, X_n)).$$

In particular, $q = (g_1(p), \dots, g_m(p))$.

Definition 5.10. The morphisms from X to Y are precisely those of the form T_h , where $h \in \text{Alg}_k(R_Y, R_X)$.

Example. Let $X = V(\mathcal{U})$, where $\mathcal{U} \subset R_{k^n} := k[X_1, \dots, X_n]$ is an ideal, and $R_X = k[x_1, \dots, x_n] := k[X_1, \dots, X_n]/\mathcal{U}$. Let $\pi : k[X_1, \dots, X_n] \rightarrow R_X$ be the quotient map. Then $\pi \in \text{Alg}_k(R_{k^n}, R_X)$ induces the inclusion $T_\pi : X \subset k^n$. (Exercise!)

Exercise. Let $X = V(y^2 - x^3) \subset k^2$. Show that the inclusion

$$k[x] \hookrightarrow \frac{k[x, y]}{(y^2 - x^3)},$$

defines the projection $X \rightarrow k$ onto the x -axis.

Definition 5.11. Two varieties X and Y are said to be biregular if \exists morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that $f \circ g$ is the identity on Y and $g \circ f$ is the identity on X .

Exercise. Let $X = V(y - x^2) \subset k^2$, and $Y = k$. Show that X and Y are biregular.

Let $X := V(\wp) \subset k^n$ be a variety, with coordinate ring R_X and quotient field $k(X) := \text{Quot}(R_X)$. $k(X)$ is called the rational function field of X .

Definition 5.12. The dimension of X is given by

$$\dim X := \text{tr deg}_k k(X).$$

Example. $\text{tr deg}_k k(X_1, \dots, X_n) = n$, hence $\dim k^n = n$.

Remark 5.13. For any algebraic set V , we have the unique decomposition $V = V_1 \cup \cdots \cup V_N$ into subvarieties. We define $\dim V = \max\{\dim V_1, \dots, \dim V_N\}$. We say V has pure dimension if $\dim V = \dim V_1 = \cdots = \dim V_N$.

Example. *Geometric form of the Normalization Lemma of Noether.* Let $X := V(\wp) \subset k^n$ be a variety of dimension d , with coordinate ring $R_X = k[x_1, \dots, x_n] = k[X_1, \dots, X_n]/\wp$. The normalization lemma asserts that one can find linear combinations

$$y_i = \sum_{j=1}^n a_{ij}x_j, \quad a_{ij} \in k, \quad i = 1, \dots, d,$$

such $\{y_1, \dots, y_d\}$ are algebraically independent over k (hence $\text{rank}(a_{ij}) = d$), and that R_X is integral over the polynomial ring $k[y_1, \dots, y_d]$ (see the Appendix for the definition of integrality). Let $\{Y_1, \dots, Y_d\}$ be variables. Then the inclusion

$$k[Y_1, \dots, Y_d] \hookrightarrow k[X_1, \dots, X_n],$$

defined by $Y_i \mapsto \sum_{j=1}^n a_{ij}X_j$ defines a surjective linear map $\pi := (a_{ij}) : k^n \rightarrow k^d$. Consider the following diagram:

$$\begin{array}{ccc} k[Y_1, \dots, Y_d] & \hookrightarrow & k[X_1, \dots, X_n] \\ \wr \downarrow & & \downarrow \\ k[y_1, \dots, y_d] & \hookrightarrow & R_X \end{array}$$

Now let $\psi \in \text{Alg}_k(k[y_1, \dots, y_d], k)$. The integrality property guarantees that ψ extends to $\tilde{\psi} \in \text{Alg}_k(R_X, k)$. It then follows that the restriction of the linear projection π to X defines a surjective morphism $\pi : X \rightarrow k^d$, whose fibers are all zero dimensional.

Example. $\dim k^n = n$. Let $f \in k[X_1, \dots, X_n]$ be a non-unit. Then $X := V(f) \subset k^n$ is called a hypersurface. If f is irreducible, then X is a variety of dimension $n - 1$. In general, X is a finite union of varieties of dimension $n - 1$.

Proposition 5.14. *Let $X \subset k^n$ be a variety and $\wp \subset R_X$ a non-zero prime ideal. Let $Y = V(\wp) \subset X$. Then $\dim Y < \dim X$.*

Proof. Let $d = \dim X$, R_X be the coordinate ring of X . Assume to the contrary. Then \exists an algebraically independent set $\{x_1, \dots, x_d\} \subset R_X$ over k , such that the images form an algebraically independent set $\{\bar{x}_1, \dots, \bar{x}_d\} \subset R_X/\wp$ over k . Let $b \in \wp$ be a nonzero element. Then $\{x_1, \dots, x_d, b\}$ cannot be algebraically independent over k . Hence \exists a polynomial $p(t, x_1, \dots, x_n) \in k[t, x_1, \dots, x_n]$ such that $P(b, x_1, \dots, x_d) = 0 \in R_X$. Since R_X is an integral domain, we may assume that P is irreducible. But P cannot be of the form $P = a \cdot t^m$ for any $m \in \mathbb{N}$, with $a \in k^\times$, as $P(b) = a \cdot b^m \neq 0$. Hence

$$P(0, \bar{x}_1, \dots, \bar{x}_d) = 0 \in R_X/\wp,$$

defines a nontrivial relation among the $\{\bar{x}_1, \dots, \bar{x}_d\}$, a contradiction. \square

Corollary 5.15. *If $X \subset k^n$ is a variety of dimension $n - 1$, then $X = V(f)$ is a hypersurface.*

Proof. Write $X = V(\wp)$, for some prime ideal $\wp \subset k[X_1, \dots, X_n]$. Since $\wp \neq (0)$ is prime, and $k[X_1, \dots, X_n]$ is a UFD, $\exists 0 \neq f \in \wp$ such that f is irreducible. Thus $X \subseteq V(f) \subsetneq k^n$. By the above proposition $\dim X = \dim V(f)$, and hence $X = V(f)$. \square

6. LOCAL STUDY OF VARIETIES

Definition 6.1. (i) For a ring \mathbb{A} , we define the Krull dimension of \mathbb{A} by:

$$\dim_{\text{Krull}} \mathbb{A} := \text{Sup}_N \{ \wp_0 \subsetneq \wp_1 \subsetneq \cdots \subsetneq \wp_N \subsetneq \mathbb{A} \},$$

where the supremum is over all ascending chains of prime ideals in \mathbb{A} .

(ii) A ring \mathbb{A} is said to be a local ring if it has only one maximal ideal. Let $\mathcal{M} \subset \mathbb{A}$ be the unique maximal ideal. $k := \mathbb{A}/\mathcal{M}$ is called the residue field. The local ring \mathbb{A} is said to be regular if $\dim_k \mathcal{M}/\mathcal{M}^2 = \dim_{\text{Krull}} \mathbb{A}$.

Remark 6.2. In the case where X/k is a classical variety, using the normalization lemma together with some standard results on integral rings (Cohen-Seidenberg, ...), this definition agrees with the previous definition of dimension, viz., $\dim X = \dim_{\text{Krull}} R_X$.

Example. Let X be a variety with coordinate ring R_X . Let $p \in X$ with maximal ideal $\mathcal{M}_p \subset R_X$. The set $S := R_X \setminus \{\mathcal{M}_p\}$ is multiplicatively closed. In particular $g \in R_X \setminus \{\mathcal{M}_p\} \Leftrightarrow g(p) \neq 0$. The localization $(R_X)_S$ is a well-known concept in commutative algebra. Define

$$\mathcal{O}_{X,p} = (R_X)_S = \left\{ \frac{f}{g} \mid g(p) \neq 0 \right\}.$$

Then $\mathcal{O}_{X,p}$ is a local ring with maximal ideal \mathcal{M}_p , where \mathcal{M}_p is identified with its image $\mathcal{O}_{X,p} \cdot \mathcal{M}_p$. Note that $\mathcal{O}_{X,p} \subset k(X)$.

Proposition 6.3. *Let X be a variety. Then*

$$R_X = \bigcap_{p \in X} \mathcal{O}_{X,p},$$

where the intersection on the RHS is in $k(X)$.

Proof. Clearly $R_X \subset \bigcap_{p \in X} \mathcal{O}_{X,p}$. Let $h \in \bigcap_{p \in X} \mathcal{O}_{X,p}$, and let $\mathcal{U} = \{g \in R_X \mid g \cdot h \in R_X\}$. Note that in $\mathcal{O}_{X,p}$, $h = f_p/g_p$ where $f_p, g_p \in R_X$ and $g_p(p) \neq 0$. Thus $g_p \in \mathcal{U}$. In particular $V(\mathcal{U}) = \emptyset$, and hence by the Nullstellensatz, $1 \in \bigcap_{p \in X} \mathcal{O}_{X,p}$. This means that $f = 1 \cdot f \in R_X$. \square

Definition 6.4. Let $X = V(\wp) \subset k^n$ be a variety and $p \in X$. The tangent space of X at p is given by

$$T_p(X) = \{p + v \in k^n \mid \nabla f(p) \bullet v = 0, \forall f \in \wp\}.$$

Exercise. Write $\wp = (f_1, \dots, f_m)$. Show that

$$T_p(X) = \{p + v \in k^n \mid \nabla f_j(p) \bullet v = 0, \forall j = 1, \dots, m\}.$$

To arrive at a coordinate invariant version of $T_p(X)$ we introduce the following.

Definition 6.5. X a variety and $p \in X$. A p -centered derivation is given by a k -linear map $D : R_X \rightarrow k$ satisfying Leibniz rule, viz.,

$$D(f \cdot g) = g(p)D(f) + f(p)D(g).$$

Remark 6.6. (i) A p -centered derivation naturally extends to a derivation on $\mathcal{O}_{X,p}$ by the quotient rule

$$D\left(\frac{f}{g}\right) = \frac{g(p)D(f) - f(p)D(g)}{g(p)^2}.$$

The p -centered derivations are denoted by $Der_p(R_X, k) = Der_k(\mathcal{O}_{X,p}, k)$.

(ii) A derivation $D : k(X) \rightarrow k(X)$ is a k -linear map satisfying Leibniz rule, viz., $D(f \cdot g) = g \cdot D(f) + f \cdot D(g) \in k(X)$. We denote this space of derivations by $Der_k(k(X), k(X))$. It is well-known that $\dim_k Der_k(k(X), k(X)) = \text{tr deg}_k k(X) = \dim X$. (This uses the fact that $k = \bar{k} \Rightarrow k(X)/k$ is separably generated.)

Exercise. Let $D \in Der_k(k(X), k(X))$. Show that $D(k) = 0$.

Proposition 6.7. *There is a canonical identification $T_p(X) \simeq Der_p(\mathcal{O}_{X,p}, k)$.*

Proof. Let $p + v \in T_p(X)$, and write $v = (\xi_1, \dots, \xi_n)$. The corresponding derivation is given by

$$D_v = \sum_{j=1}^n \xi_j \frac{\partial}{\partial x_j} \Big|_p.$$

Going the other way, let $D \in Der_p(R_X, k)$. Via the quotient map

$$k[X_1, \dots, X_n] \rightarrow R_X = \frac{k[X_1, \dots, X_n]}{\wp},$$

it is clear that D extends to a p -centered derivation on $k[X_1, \dots, X_n]$ that kills both \wp and k . For $f \in \wp$, we can write

$$f = f(p) + \sum_{j=1}^n \frac{\partial f}{\partial x_j}(p)(x_j - p_j) + \sum_{\alpha} g_{\alpha} \cdot h_{\alpha},$$

where $g_{\alpha}(p) = h_{\alpha}(p) = 0$. Leibniz rule implies that $D(g_{\alpha} \cdot h_{\alpha}) = 0$. Then if we set $\xi_j = D(x_j) \in k$, we have

$$0 = D(f) = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(p) \cdot \xi_j = \sum_{j=1}^n \xi_j \frac{\partial}{\partial x_j} \Big|_p (f).$$

Hence $D = D_v$, where $v = (\xi_1, \dots, \xi_n)$ and $p + v \in T_p(X)$. \square

Now assume given varieties $X \subset k^n$, $Y \subset k^m$ and a morphism $T_h : X \rightarrow Y$ (induced by some $h \in \text{Alg}_k(R_Y, R_X)$ with $q = T_h(p)$). Let $D \in Der_p(\mathcal{O}_{X,p}, k)$ and $g \in \mathcal{O}_{Y,q}$ be given. Note that $g \circ T_h \in \mathcal{O}_{X,p}$. [In fact $g \circ T_h = h(g)$.] Define $dT_h(D)(g) := D(g \circ T_h) \in k$. This defines a linear map (called the differential of T_h),

$$dT_h(p) : T_p(X) \rightarrow T_q(Y).$$

Exercise. Express $dT_h(p)$ in terms of the jacobian matrix of a map from $k^n \rightarrow k^m$, sending $T_p(X) \subset k^n$ to $T_q(Y) \subset k^m$.

Exercise. Let $X = V(y^2 - x^3) \subset k^2$. Show that X and k are not biregular.

Exercise. Let $\mathcal{M}_p \subset \mathcal{O}_{X,p}$ be the (unique) maximal ideal. Show that any $D \in Der_p(\mathcal{O}_{X,p}, k)$ kills \mathcal{M}_p^2 . Deduce that

$$T_p(X) \simeq \text{hom}_k\left(\frac{\mathcal{M}_p}{\mathcal{M}_p^2}, k\right).$$

Definition 6.8. The space of differentials at $p \in X$ is given by the cotangent space $T_p(X)^* = \mathcal{M}_p / \mathcal{M}_p^2$.

Now let $X = V(\wp) \subset k^n$ be a variety, with coordinate ring R_X and rational function field $k(X)$. We want to show that $\dim_k T_p(X) \geq \dim X$ with equality on a non-empty Zariski open subset of X . We recall that $\text{trdeg}_k k(X) = \text{dimension of the } k\text{-vector space of derivations on } k(X) \text{ killing } k$. Write $\wp = (f_1, \dots, f_m)$.

$$\begin{aligned} \text{tr deg}_k k(X) &= \dim_k \text{Der}_k(k(X), k(X)) = \dim_k \text{Der}_k(R_X, k(X)) \\ &= \dim_k \{D \in \text{Der}_k(R_{k^n}, k(X)) \mid D\wp = 0\} \end{aligned}$$

Now for $f \in R_{k^n}$ and $D \in \text{Der}_k(R_{k^n}, k(X))$,

$$Df = \sum_{j=1}^n \frac{\partial f}{\partial x_j} D(x_j), \quad D(x_j) \in k(X).$$

Thus

$$\begin{aligned} \text{tr deg}_k k(X) &= \dim_{k(X)} \ker \left[\begin{pmatrix} \frac{\partial f_i}{\partial x_j} \end{pmatrix} : k(X)^n \rightarrow k(X)^m \right] \\ &= n - \text{rank} \left(\frac{\partial f_i}{\partial x_j} \right). \end{aligned}$$

So let

$$r = \text{rank} \left(\frac{\partial f_i}{\partial x_j} \right).$$

Then $\exists A \in GL(n, k(X))$ and $B \in GL(m, k(X))$ such that

$$A \left(\frac{\partial f_i}{\partial x_j} \right) B = \begin{pmatrix} I_r & | & 0 \\ \hline 0 & | & 0 \end{pmatrix}$$

Collecting all the R_X -denominators of the $k(X)$ -coefficients in A , B , and multiplying them together to get a $g \in R_X$, it follows that on the open set X_g ,

$$p \in X_g \Rightarrow \text{rank} \left(\frac{\partial f_i}{\partial x_j} \right)(p) = \text{rank} \left(\frac{\partial f_i}{\partial x_j} \right),$$

i.e.

$$p \in X_g \Leftrightarrow \dim_k T_p(X) = \dim X.$$

Exercise. Show that $\{p \in X \mid \dim T_p \geq \ell\}$ is a Zariski closed subset of X . [Hint: Look at the ideal of $(n - \ell) \times (n - \ell)$ minors of $\left(\frac{\partial f_i}{\partial x_j} \right)$.] So for example, $X = \{p \in X \mid \dim_k T_p(X) \geq \dim X\}$, and if we put $\text{Sing}(X) = \{p \in X \mid \dim_k T_p(X) > \dim X\}$, then this is a proper Zariski closed subset of X . In particular $\dim \text{Sing}(X) < \dim X$.

Tangent cone. Let $X = V(\wp) \subset k^n$ be a variety with prime ideal \wp . Let $p = (p_1, \dots, p_n) \in X$. Any $f \in \wp \subset k[X_1 - p_1, \dots, X_n - p_n] = k[X_1, \dots, X_n]$ has a Taylor expansion with no constant term. The leading homogeneous term will be denoted by f_h . The (homogeneous) linear term of f is denoted by f_L . Let \mathcal{U}_L be the ideal generated by the linear terms $(f_L \mid f \in \wp)$, and \mathcal{U}_h the ideal generated by $(f_h \mid f \in \wp)$. Note that $T_p(X) = V(\mathcal{U}_L)$ and we now define the tangent cone to X at $p \in X$ by $C_p(X) = V(\mathcal{U}_h)$. Since $\mathcal{U}_L \subseteq \mathcal{U}_h$, it is clear that $C_p(X) \subseteq T_p(X)$. One can interpret $T_p(X)$ as the algebraic linear hull of $C_p(X)$. In the case $k = \mathbb{C}$,

$C_p(X)$ is described as the limiting secants to X at p . Such a description enables one to prove that $\dim C_p(X) = \dim X$. In fact it is known that $C_p(X)$ has pure dimension $= \dim X$. Thus another way of interpreting p as nonsingular, is to say the inclusion $C_p(X) \subseteq T_p(X)$ is an equality.

Exercise. Let $X = V(f(x, y)) \subset k^2$, and $p \in X$. Show that $C_p(X)$ is a union of lines in k^2 passing through p .

Definition 6.9. Let X be a variety. The proper subvariety $\text{Sing}(X) \subset X$ is called the singular set of X . The open subset $X_{\text{sm}} \subset X$ is called the smooth or nonsingular part of X . X is said to be a smooth or nonsingular variety if $\text{Sing}(X) = \emptyset$.

Exercise. Show that any variety X can be stratified into a finite disjoint union of locally closed smooth varieties (locally closed means Zariski open in its closure). [Hint: First write $X = X_{\text{sm}} \amalg \text{Sing}(X)$. Then decompose $\text{Sing}(X)$ into a finite union of varieties, and continue decomposing. This process must terminate because of decreasing dimensions of varieties.]

7. AFFINE SCHEMES

Let \mathbb{A} be a commutative ring with unity. Recall that $X := \text{Spec}(\mathbb{A}) = \{\text{prime ideals in } \mathbb{A}\}$. The closed points of X are $\text{Max}(\mathbb{A})$. Let $E \subset \mathbb{A}$ be a subset. We define

$$V(E) = \{\wp \in \text{Spec}(\mathbb{A}) \mid E \subset \wp\}.$$

Let \mathcal{U} be the ideal generated by E . Since $\bigcap_{\wp \supset E} \wp$ is an ideal, we deduce that $V(E) = V(\mathcal{U}) = V(\sqrt{\mathcal{U}})$. Observe that $V(\sum_{\alpha \in I} \mathcal{U}_\alpha) = \bigcap_{\alpha \in I} V(\mathcal{U}_\alpha)$, and that $V(\mathcal{U}_1 \cdot \mathcal{U}_2) = V(\mathcal{U}_1) \cup V(\mathcal{U}_2)$. Note that $\text{Spec}(\mathbb{A}) = V(0) = V(\mathcal{N}(\mathbb{A}))$, where $\mathcal{N}(\mathbb{A})$ is the nilradical of \mathbb{A} (see the Appendix), and that $V(1) = \emptyset$. In particular, we have the Zariski topology on $\text{Spec}(\mathbb{A})$. The beauty behind $\text{Spec}(\mathbb{A})$ is its apparent functoriality with respect to commutative rings. If $f: \mathbb{A} \rightarrow \mathbb{B}$ is a ring homomorphism (with $f(1) = 1$), then $f^{-1}(\text{prime}) = \text{prime}$. In particular we have $f^*: \text{Spec}(\mathbb{B}) \rightarrow \text{Spec}(\mathbb{A})$. To arrive at something similar on the level of maximal ideals, requires restricting to k -algebras of finite type, where $k = \bar{k}$, using the Nullstellensatz. For example the inclusion $f: \mathbb{Z} \hookrightarrow \mathbb{Q}$ has the property that $f^{-1}(0) = (0)$. Note that $(0) \subset \mathbb{Q}$ is maximal; however $(0) \subset \mathbb{Z}$ is prime, but not maximal. If $X = \text{Spec}(\mathbb{A})$, then $\dim X := \text{Krull dimension of } \mathbb{A}$. As we mentioned earlier, in the case where X/k is a classical variety, this coincides with the previous definition of dimension in terms of transcendence degree.

Exercise. Define $I(V(\mathcal{U}))$ where $\mathcal{U} \subset \mathbb{A}$ is an ideal. Explain why $I(V(\mathcal{U})) = \sqrt{\mathcal{U}}$.

Exercise. Suppose that \mathbb{A} is an integral domain. The prime ideal $(0) \in X := \text{Spec}(\mathbb{A})$ is called the generic point of X . Show that the Zariski closure $\overline{\{(0)\}} = X$.

Exercise. Let $X = \text{Spec}(\mathbb{A})$, $f \in \mathbb{A}$, and let $X_f = X \setminus V(f)$. Show that $\{X_f \mid f \in \mathbb{A}\}$ forms a basis for the Zariski topology of X .

Exercise. Let $f \in \mathbb{A}$ and put $S = \{1, f^2, f^3, \dots\}$. The localization of \mathbb{A} with respect to S , denoted by $\mathbb{A}_{(f)}$, is defined in any commutative algebra text. Show that $\text{Spec}(\mathbb{A}_{(f)}) = X_f$.

Exercise. Let $\wp \in X := \text{Spec}(\mathbb{A})$, and $S = \mathbb{A} \setminus \wp$. Put $\mathcal{O}_{X,\wp} = \mathbb{A}_S$. Show that $\mathcal{O}_{X,\wp}$ is a local ring with maximal ideal $\mathcal{M} := \mathcal{O}_{X,\wp} \cdot \wp$. The field $k := \mathcal{O}_{X,\wp}/\mathcal{M}$ is called the residue field. $\wp \in X$ is called a regular point if $\dim_k \mathcal{M}/\mathcal{M}^2 = \dim \mathcal{O}_{X,\wp}$.

Definition 7.1. $X := \text{Spec}(\mathbb{A})$ is said to be regular if every point in X is a regular point.

Exercise. Show that $X_f \cap X_g = X_{fg}$.

Exercise. Let $\mathcal{U} \subset \mathbb{A}$ be an ideal. Show that $V(\mathcal{U}) = \emptyset \Leftrightarrow \mathcal{U} = (1)$.

Exercise. Let $X = \text{Spec}(\mathbb{A})$. Show that X is irreducible $\Leftrightarrow \mathbb{A}$ is an integral domain.

Exercise. Show that $\text{Spec}(\mathbb{A} \oplus \mathbb{B}) = \text{Spec}(\mathbb{A}) \amalg \text{Spec}(\mathbb{B})$.

Exercise. Consider two homomorphisms of rings $C \rightarrow A, C \rightarrow B$. The push-out:

$$\begin{array}{ccc} & A \otimes_C B & \\ & \nearrow & \nwarrow \\ A & & B \\ & \nwarrow & \nearrow \\ & C & \end{array}$$

translates to the fiber product $\text{Spec}(A) \times_{\text{Spec}(C)} \text{Spec}(B) := \text{Spec}(A \otimes_C B)$ on the level of affine schemes. Show that if A, B, C are k -algebras of finite type over $k = \bar{k}$, then $\text{Spec}(A) \times_{\text{Spec}(C)} \text{Spec}(B)$ is bijective to the corresponding fiber product in the category of sets.

Remark 7.2. Let K/k be a Galois extension of k of degree $m > 1$. Then

$$K \otimes_k K \simeq \underbrace{K \oplus \cdots \oplus K}_{m \text{ times}},$$

the isomorphism given by

$$a \otimes b \mapsto (ab_1, \dots, ab_m),$$

where $\{b_1, \dots, b_m\} = \{\sigma(b) \mid \sigma \in \text{Gal}(K/k)\}$. Hence $\text{Spec}(K \otimes_k K)$ is not irreducible. In particular, the category of irreducible affine schemes is not closed under products.

8. EXAMPLES OF AFFINE SCHEMES

(i) $X = \text{Spec}(\mathbb{Z}) = \underbrace{\coprod_{p \in \mathbb{N} \text{ prime}} \{(p)\}}_{\text{closed points}} \amalg \underbrace{\{(0)\}}_{\text{generic point}}$. $(0) \subsetneq (p) \Rightarrow \dim X = 1$.

(ii) k a field. $X = \text{Spec}(k) = \{(0)\}$, $\dim X = 0$.

(iii) k a field.

$$X := \text{Spec}(k[x]) = \underbrace{\{(f) \mid f \text{ irreducible}\}}_{\text{closed points}} \amalg \underbrace{\{(0)\}}_{\text{generic point}}, \quad \dim X = 1.$$

In the case $k = \bar{k}$

$$X = \{(x - a) \mid a \in k\} \amalg \{(0)\} \simeq k \amalg \{(0)\} \quad (= \text{affine line})$$

(iv) For simplicity, assume $k = \bar{k}$. Let $X = \text{Spec}(k[x, y])$ (affine plane). The inclusions

$$(0) \subsetneq (x) \subsetneq (x, y),$$

implies that $\dim X = 2$. We have

$$X = \begin{cases} (0) & \text{generic point, (codim 2).} \\ (f) & f \text{ irreducible, (codim 1).} \\ (x - a, y - b) & (a, b) \in k^2, \text{ (codim 2, closed points).} \end{cases}$$

(v) Let $\mathbb{A} = \mathbb{R}[x]/(x^2 + 1)$. Then $\text{Spec}(\mathbb{A})$ is irreducible. However if we perform a base change $\mathbb{A}_{\mathbb{C}} := \mathbb{A} \otimes_{\mathbb{R}} \mathbb{C}$, we have $\text{Spec}(\mathbb{A}_{\mathbb{C}})$ reducible. In this situation say that $\text{Spec}(\mathbb{A})$ is not geometrically irreducible.

(vi) $X = \text{Spec}(\mathbb{Z}[x])$, $\dim X = 2$. X consists of the following:

(a) The generic point (0)

(b) Principal prime ideals (f) where either $f = p$ is a prime in \mathbb{Z} , or f is a $\mathbb{Q}[x]$ irreducible polynomial such that its coefficients have $\text{GCD} = 1$.

(c) Maximal ideals (p, f) , where $p \in \mathbb{Z}$ is prime and f is a monic polynomial, irreducible in $\mathbb{Z}_p[x]$.

9. APPENDIX: THE TWO HILBERT THEOREMS

(I) The basis theorem. Let \mathbb{A} be a commutative ring with unity $1 \in \mathbb{A}$.

Lemma 9.1. *Let M be an \mathbb{A} -module. TFAE:*

- (i) *Every increasing sequence of submodules in M is stationary.*
- (ii) *Every (non-empty) subset of submodules of M has a maximal element.*
- (iii) *Every submodule is finitely generated.*

Proof. If $N \subset M$ were not finitely generated, then \exists sequence $\{f_n\}_{n \in \mathbb{N}} \subset N$ such that

$$\mathbb{A} \cdot f_1 \subsetneq \mathbb{A} \cdot (f_1, f_2) \subsetneq \mathbb{A} \cdot (f_1, f_2, f_3) \subsetneq \cdots,$$

which is incompatible with (i). Therefore (i) \Rightarrow (iii). On the other hand, if we have an increasing sequence of submodules of M ,

$$M_1 \subset M_2 \subset M_3 \subset \cdots,$$

then set $N = \bigcup_{j \in \mathbb{N}} M_j$. If N is finitely generated, then we have $N = \mathbb{A} \cdot \{f_1, \dots, f_r\}$ where $f_i \in M_{j_i}$. Thus $M_j = M_{j+1} = \cdots$, where $j \geq \max\{j_1, \dots, j_r\}$. This shows that (iii) \Rightarrow (i). The equivalence (i) \Leftrightarrow (ii) is left to the reader. \square

Definition 9.2. An \mathbb{A} -module M is said to be Noetherian if it satisfies (i)-(ii)-(iii) above. The ring \mathbb{A} is Noetherian, if it is a Noetherian \mathbb{A} -module, i.e. every ideal is finitely generated (equivalently every ascending chain of ideals is stationary).

Lemma 9.3. *Let $N \subset M$ be \mathbb{A} -modules. Then M is Noetherian $\Leftrightarrow M$ & M/N are Noetherian.*

Proof. This is an exercise! \square

Corollary 9.4. \mathbb{A} is Noetherian $\Leftrightarrow \mathbb{A}^m$ is Noetherian $\forall m \in \mathbb{N}$.

Theorem 9.5 (Basis theorem). *If \mathbb{A} is Noetherian, then so is $\mathbb{A}[t]$.*

Proof. Let $\mathcal{U} \subset \mathbb{A}$ be an ideal. We will show that \mathcal{U} is finitely generated. Let $\mathcal{U}_0 \subset \mathbb{A}$ be the ideal of leading coefficients of elements in \mathcal{U} . By assumption $\mathcal{U}_0 = (a_1, \dots, a_n)$ is finitely generated. For each a_i , we pick an $f_i \in \mathcal{U}$ of the form $f_i = a_i t^r + \text{lower order terms}$, where we can insist that r is independent of i , [e.g. If $\tilde{f}_i = a_i t^{r_i} + \dots$, put $f_i = t^{r-r_i} \cdot \tilde{f}_i$, where $r = \max\{r_i\}$.] Let $\mathcal{U}' = (f_1, \dots, f_n) \subset \mathcal{U}$, and let $f \in \mathcal{U}$. Then $f = at^m + \dots$, where $a \in \mathcal{U}_0$. If $m > r$, consider $g := f - \sum_i c_i f_i t^{m-r}$, where $a = \sum_i c_i a_i$. Obviously $\deg g < \deg f$. By repeating this process (if necessary), we can assume (by relabeling) given $g \in \mathcal{U}'$ such that $f = g + h$, where $\deg h \leq r$. But $h \in \mathcal{U} \cap B$, where $B \subset \mathbb{A}[t]$ is the \mathbb{A} -module generated by $\{1, t, \dots, t^r\}$. But $B \simeq \mathbb{A}^{r+1}$ is a Noetherian \mathbb{A} -module, and hence $\mathcal{U} \cap B$ is finitely generated, being an \mathbb{A} -submodule of B . Let us write $B \cap \mathcal{U} = \mathbb{A} \cdot (h_1, \dots, h_m)$. We have shown that

$$\mathcal{U} = \mathbb{A} \cdot (f_1, \dots, f_n, h_1, \dots, h_m),$$

which is finitely generated. \square

(II) Notes on integrality. Let \mathbb{A} be a ring. We say that an \mathbb{A} -module M is faithful if the annihilator ideal $\text{Ann}(M) := \{a \in \mathbb{A} \mid a \cdot M = 0\} = \{0\}$ is zero.

Proposition 9.6. *Let $\mathbb{A} \subset \mathbb{B}$ be rings, and $\xi \in \mathbb{B}$. TFAE:*

- (i) \exists monic polynomial $P(t) \in \mathbb{A}[t]$ such that $P(\xi) = 0 \in \mathbb{B}$.
- (ii) $\mathbb{A}[\xi]$ is a finitely generated \mathbb{A} -module.
- (iii) \exists a faithful finitely generated $\mathbb{A}[\xi]$ -module M which is finitely generated over \mathbb{A} .

Proof. (i) \Rightarrow (ii): If $P(t) = t^r + a_{r-1}t^{r-1} + \dots + a_0 \in \mathbb{A}[t]$, then $P(\xi) = 0 \Rightarrow \mathbb{A}[\xi]$ is generated by $\{1, \xi, \dots, \xi^{r-1}\}$.

(ii) \Rightarrow (iii): Set $M = \mathbb{A}[\xi]$. Then $1 \in M$, hence $\text{Ann}(M) = 0$.

(iii) \Rightarrow (i): Assume given $\mathbb{A}[\xi]$ -faithful M with generators $\{u_1, \dots, u_m\}$ over \mathbb{A} . Then $\xi \cdot M \subset M$ implies that

$$\xi \cdot \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} = (a_{ij}) \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix},$$

for some $m \times m$ matrix (a_{ij}) , $a_{ij} \in \mathbb{A}$. Thus

$$(\xi \cdot I_m - (a_{ij})) \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} = 0.$$

Cramer's rule tells us that \exists a $m \times m$ matrix Q with coefficients in $\mathbb{A}[\xi]$ such that $Q \cdot (I_m - (a_{ij})) = \det(I_m - (a_{ij})) \cdot I_m$. Thus $\det(\xi \cdot I_m - (a_{ij})) u_j = 0 \forall j$, hence $\det(I_m - (a_{ij})) \cdot M = 0$. By faithfulness, $\det(\xi \cdot I_m - (a_{ij})) = 0$. Now put $P(t) = \det(t \cdot I_m - (a_{ij}))$. \square

Let us assume the setting of the above proposition.

Definition 9.7. $\xi \in \mathbb{B}$ is said to be integral over \mathbb{A} if it satisfies any of the equivalent statements in (i), (ii), and (iii) above.

Proposition 9.8. Let $C(\mathbb{A}, \mathbb{B}) := \{\xi \in \mathbb{B} \mid \xi \text{ is integral over } \mathbb{A}\}$. Then $C(\mathbb{A}, \mathbb{B}) \subset \mathbb{B}$ is a subring, called the integral closure of \mathbb{A} in \mathbb{B} .

Proof. Let $b_1, b_2 \in \mathbb{B}$ be integral over \mathbb{A} . Then $\mathbb{A}[b_1, b_2]$ is a finitely generated $\mathbb{A}[b]$ -module, where $b = b_1 + b_2$ or $b = b_1 \cdot b_2$. \square

Exercise. Let \mathbb{A} be a UFD and $K = \text{Quot}(\mathbb{A})$ the quotient field. Show that \mathbb{A} is integrally closed in K (i.e. \mathbb{A} is the integral closure of \mathbb{A} in K).

(III) The Nullstellensatz. In this part, we are going to assume that k is any field, with algebraic closure \bar{k} . For any commutative ring \mathbb{A} with 1, we recall the following subrings:

Definition 9.9. (i) The nilradical of \mathbb{A} is given by the intersection of all its prime ideals, viz.,

$$\mathcal{N}(\mathbb{A}) := \bigcap_{\wp \subset \mathbb{A}} \wp.$$

(ii) The Jacobson radical of \mathbb{A} is given by the intersection of all its maximal ideals, viz.,

$$\mathcal{R}(\mathbb{A}) := \bigcap_{\mathcal{M} \subset \mathbb{A}} \mathcal{M}.$$

Observe that $\mathcal{N}(\mathbb{A}) \subseteq \mathcal{R}(\mathbb{A})$. Now let $(1) \neq \mathcal{U} \subset k[X_1, \dots, X_n]$ be any ideal, and put

$$\mathbb{A} := \frac{k[X_1, \dots, X_n]}{\mathcal{U}} = k[x_1, \dots, x_n],$$

which we recall is a k -algebra of finite type. Let $\pi : k[X_1, \dots, X_n] \rightarrow \mathbb{A}$ be the quotient map. Observe that

$$\pi^{-1}(\mathcal{N}(\mathbb{A})) = \bigcap_{\wp \supset \mathcal{U}} \wp.$$

Lemma 9.10. Let $S \subset \mathbb{A}$ be a multiplicatively closed set with $0 \notin S$. Then any maximal element of

$$\Sigma := \{\text{Ideals } \mathcal{U} \subset \mathbb{A} \mid \mathcal{U} \cap S = \emptyset\},$$

is prime.

Proof. Let \wp be maximal (\exists by Zorn's lemma). Take $a, b \in \mathbb{A}$ with $a, b \notin \wp$. Then $\wp + (a)$ and $\wp + (b)$ both intersect S . So write $s_1 = x_1 + at_1$, $s_2 = x_2 + bt_2$ with $s_1, s_2 \in S$, $x_1, x_2 \in \wp$, $t_1, t_2 \in \mathbb{A}$. Suppose that $ab \in \wp$. Then

$$S \ni s_1 s_2 = x_1 x_2 + x_1 b t_2 + a t_1 x_2 + t_1 t_2 a b \in \wp,$$

i.e. $\wp \cap S \neq \emptyset$, which cannot happen. Hence $ab \notin \wp$ and therefore \wp must be prime. \square

Definition 9.11. $a \in \mathbb{A}$ is nilpotent if $a^m = 0$ for some $m = m(a) \in \mathbb{N}$. Equivalently, a is nilpotent $\Leftrightarrow a \in \sqrt{(0)}$.

Corollary 9.12. $\mathcal{N}(\mathbb{A}) = \sqrt{(0)}$, hence

$$\sqrt{\mathcal{U}} = \bigcap_{\wp \supset \mathcal{U}} \wp.$$

Proof. It suffices to show that $a \in \mathcal{N}(\mathbb{A}) \Rightarrow a$ is nilpotent. Assume to the contrary, and put $S = \{a^n \mid n \in \mathbb{N}\}$. Then $0 \notin S$, and hence \exists a prime \wp such that $\wp \cap S = \emptyset$. Hence $a \notin \wp$ and so $a \notin \mathcal{N}(\mathbb{A})$. \square

The Nullstellensatz takes the following equivalent forms:

Proposition 9.13. *Let \mathbb{A} be a k -algebra of finite type. The following statements are equivalent:*

(i) *Let $\xi \in \mathbb{A}$ with $\xi \notin \mathcal{N}(\mathbb{A})$. Then $\exists \psi \in \text{Alg}_k(\mathbb{A}, \bar{k})$ such that $\psi(\xi) \neq 0$.*

(ii) $\mathcal{N}(\mathbb{A}) = \mathcal{R}(\mathbb{A})$.

(iii) *Let $f, f_1, \dots, f_r \in k[X_1, \dots, X_n]$ be given. If $f(p) = 0 \forall p \in \bar{k}^n$ which satisfy $f_1(p) = \dots = f_r(p) = 0$, then for some $m \in \mathbb{N}$, $f^m = g_1 f_1 + \dots + g_r f_r$ for some $g_1, \dots, g_r \in k[X_1, \dots, X_n]$. [This is the same thing as saying $I(V(\mathcal{U})) = \sqrt{I}$, with $K = \bar{k}$.]*

(iv) *If \mathbb{A} is a field, then \mathbb{A} is algebraic, i.e. \exists an embedding $\mathbb{A} \hookrightarrow \bar{k}$ over k . [Thus \mathbb{A} is a field $\Leftrightarrow \mathbb{A}$ is algebraic.]*

Proof. The equivalence of (i), (ii) and (iii) is an exercise. To connect (iv) with (i), consider the following. If $\xi \in \mathbb{A}$ with $\xi \notin \mathcal{N}(\mathbb{A})$, then \exists a prime $\wp \subset \mathbb{A}$ such that $\xi \notin \wp$. By replacing \mathbb{A} by \mathbb{A}/\wp , we may assume that \mathbb{A} is an integral domain and that $\xi \in \mathbb{A} \setminus \{0\}$. Now put

$$\mathbb{A}' := \frac{\mathbb{A}[t]}{(t\xi - 1)}.$$

Note that \mathbb{A}' is a k -algebra of finite type. Let $\mathcal{M} \subset \mathbb{A}'$ be any maximal ideal (which exists by Zorn's lemma). If we assume (iv), then $\mathbb{A}'/\mathcal{M} \hookrightarrow \bar{k}$, i.e. $\exists \psi' \in \text{Alg}_k(\mathbb{A}', \bar{k})$ such that $\ker \psi' = \mathcal{M}$. But $\xi \in \mathbb{A}'^*$ is a unit. Hence $\psi'(\xi) \neq 0$. Now put $\psi = \psi'|_{\mathbb{A}} \in \text{Alg}_k(\mathbb{A}, \bar{k})$. Hence (iv) \Rightarrow (i). It is easy to see how (iii) \Rightarrow (iv) from Corollary 5.2, and is left as an exercise. \square

Theorem 9.14 (Nullstellensatz). *Let B be of finite type over a field k . If B is a field, then B is a finite algebraic extension of k .*

Proof. Write $B = k[x_1, \dots, x_n]$. We will argue by induction on n . The case $n = 1$ is obvious. If we put $A = k[x_1]$, then $B = A[x_2, \dots, x_n]$. Since B is a field, $A \subset B$ must be an integral domain. Let $K = \text{Quot}(A)$. Since B is a field, and $A \subset B$, we must have $K \subset B$ as well. Thus $B = K[x_2, \dots, x_n]$ is a K -algebra of finite type. By induction on n , B is algebraic over K . Thus we are done if we can show that x_1 is algebraic over k . Assume to the contrary that x_1 is transcendental over k . Then A is a ring of polynomials, hence a UFD. Let $f \in A$ be the common denominator for the monic coefficient polynomials that occur in the equations for $\{x_2, \dots, x_n\}$ over K . Put $S = \{f, f^2, f^3, \dots\}$ which is a multiplicative set. One can form a ring $A_S \subset K$ consisting of elements of the form $\{a/b \in K \mid a \in A \text{ \& } b \in S\}$. Since A is a UFD, then so is A_S (why?). Since each x_i is the root of a monic polynomial equation with coefficients in A_S , it follows that B is integral over A_S . In particular, $K \subset B$ is integral over A_S . Since A_S , being a UFD, is integrally closed in $K = \text{Quot}(A_S) = \text{Quot}(A)$, it follows that $A_S = K$, which is nonsense (why?). Therefore x_1 cannot be transcendental over k , and we are done. \square

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